Linear systems – Final exam – Version B

Final exam 2020–2021, Tuesday 15 June 2021, 15:00 – 18:30

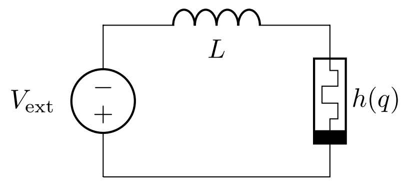
Instructions

1. The exam is **open book**, meaning that the use of the lecture notes and other course material (everything you can find on the Nestor page of the course) is allowed. You are also allowed to use your own handwritten notes.
2. The use of sources other than described above, such as books or web pages, is **not** allowed.
3. All answers need to be accompanied with an explanation or calculation.
4. Please clearly indicate the exam version on your exam paper.
5. Please handwrite your solutions, photograph your exam paper and submit your work as a single pdf file.

The exam comprises 5 problems on the next two pages.

Problem 1

(3 + 7 + 4 + 10 + 4 = 28 points)



Electrical circuits with nonlinear elements are expected to form the building blocks for future brain-inspired computers. The simplest such circuit, depicted above, can be modelled as

$$\begin{aligned} L\dot{I}(t) &= -\frac{\mathrm{d}h}{\mathrm{d}q}\big(q(t)\big)I(t) + V_{\text{ext}}(t), \\ \dot{q}(t) &= I(t), \end{aligned} \tag{1}$$

where $I(t) \in \mathbb{R}$ is the current through the inductor with inductance $L > 0$. The nonlinear element has the internal state variable $q(t) \in \mathbb{R}$ and is characterized by the smooth function h satisfying

$$\frac{\mathrm{d}h}{\mathrm{d}q}(q) > 0 \quad \text{for all } q \in \mathbb{R}.$$

Finally, $V_{\text{ext}}(t) \in \mathbb{R}$ is the external voltage applied to the circuit.

(a) The total energy in the circuit is given by the function $E(I) = \frac{1}{2}LI^2$. Let $V_{\text{ext}}(t) = 0$ for all t. Show that the circuit dissipates energy, i.e., the total energy $E(I(t))$ is not increasing as a function of time.

(b) Show that, for any $\bar{q} \in \mathbb{R}$, $(I, q) = (0, \bar{q})$ is an equilibrium point for the constant input $V_{\text{ext}}(t) = \bar{V}_{\text{ext}}$ with $\bar{V}_{\text{ext}} = 0$. Moreover, linearize the dynamics (1) around the equilibrium $(0, \bar{q})$ for $\bar{V}_{\text{ext}} = 0$.

(c) Using the linearization in (b), show that the equilibrium $(0, \bar{q})$ for $\bar{V}_{\text{ext}} = 0$ is not asymptotically stable.

In the remainder of this problem, we consider the initial value problem (1) with initial conditions

$$I(0) = I_0, \qquad q(0) = q_0. \tag{2}$$

and $V_{\text{ext}}(t) = 0$ for all $t \geq 0$.

(d) By explicitly solving the first equation in (1), show that the initial value problem (1), (2) can be written in the simpler form

$$\dot{q}(t) = I_0 - \frac{1}{L}\big(h(q(t)) - h(q_0)\big), \qquad q(0) = q_0. \tag{3}$$

In addition, show how $I(t)$ can be obtained from a solution $q(t)$ to (3).

(e) It can easily be shown that the dynamics in (3) has a *unique* equilibrium point q^* and that the solution to the initial value problem (3) satisfies

$$\lim_{t\to\infty} q(t) = q^*.$$

Why does this not contradict the answer in (c)?

Problem 2

(15 points)

Consider the linear system given by the transfer function

$$T(s) = \frac{s+2}{s^4 + as^3 + 4as^2 + 2as + 3a}$$

with $a \in \mathbb{R}$. Give the values of a for which the system is externally stable.

Problem 3

(4 + 10 + 8 = 22 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with state $x(t) \in \mathbb{R}^2$, input $u(t) \in \mathbb{R}$, and matrices

$$A = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(a) Verify that the system is controllable.

(b) Find a nonsingular matrix T and real numbers α_1, α_2 such that

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \qquad TB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hint. It is sufficient to give T^{-1}.

(c) Use the matrix T from (b) to design a state feedback controller $u(t) = Fx(t)$ such that the closed-loop system matrix $A + BF$ has eigenvalues -1 and -2.

Problem 4

(5 + 5 = 10 points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ -4 & 2 & 1 \end{bmatrix} x(t), \qquad y(t) = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} x(t).$$

(a) Show that the system is not observable and give a basis for the unobservable subspace.

(b) Is the system detectable?

Problem 5

(15 points)

Consider the linear system

$$\dot{x}(t) = Ax(t), \qquad y(t) = Cx(t) \tag{4}$$

with $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$. The linear system is called *output stable* if

$$\lim_{t\to\infty} Ce^{At}x_0 = 0 \quad \text{for all } x_0 \in \mathbb{R}^n.$$

Show that the following two statements are equivalent:

1. (4) is output stable;
2. every eigenvalue λ of A satisfying $\mathrm{Re}(\lambda) \geq 0$ is not (A, C)-observable.

Hint. You may use the following fact: for an eigenvector v corresponding to an eigenvalue λ of A, we have that $e^{At}v = ve^{\lambda t}$.

Note. The implication 2. $\Rightarrow$ 1. does not hold, as mentioned during the exam. Grading has been adapted.

(10 points free)